To obtain an equivalent expression using the flexibility coefficients from Eq. (3), we write

$$\begin{Bmatrix} U_s \\ U_o \end{Bmatrix} = \begin{bmatrix} E_{ss} & E_{so} \\ E_{os} & E_{oo} \end{bmatrix} \begin{Bmatrix} P_s \\ O \end{Bmatrix} + \begin{bmatrix} \Phi_{sr} \\ \Phi_{or} \end{bmatrix} \{U_r\} \tag{7}$$

where Φ_{sr} and Φ_{or} are the rigid body mode shapes for the s d.o.f. and the o d.o.f., respectively.

Expand Eq. (7) to obtain

$$U_s = E_{ss} P_s + \Phi_{sr} U_r \tag{7a}$$

$$U_o = E_{os} P_s + \Phi_{or} U_r \tag{7b}$$

Multiply Eq. (7a) by E_{ss}^{-1} and solve for P_s; then substitute into Eq. (7b).

$$U_o = E_{os} E_{ss}^{-1} U_s + (\Phi_{or} - E_{os} E_{ss}^{-1} \Phi_{sr}) U_r$$

The transformation, compatible with Eq. (6), is

$$\begin{Bmatrix} U_s \\ U_r \\ U_o \end{Bmatrix} = \begin{bmatrix} I & O \\ O & I \\ E_{os} E_{ss}^{-1} & (\Phi_{or} - E_{os} E_{ss}^{-1} \Phi_{sr}) \end{bmatrix} \begin{Bmatrix} U_s \\ U_r \end{Bmatrix} \tag{8}$$

The transformation of Eq. (8) can be used to reduce the mass matrix. This method requires only the rigid body modes in addition to the deflection matrices, E_{os} and E_{ss}, and the inverse matrix E_{ss}^{-1}. The "s" set is much smaller than the "o" set, therefore, this method should be faster and less expensive than the Guyan reduction, since it is not necessary to invert K_{oo}.

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Design Plastic Stress Concentration Factors Using Ramberg-Osgood Stress-Strain Parameters

Ralph Papirno*

Army Materials and Mechanics Research Center, Watertown, Mass.

Nomenclature

E = modulus of elasticity
K_t = theoretical elastic stress or strain concentration factor
K^* = limit value of K_t for elastic reference stress
K_σ = plastic stress concentration factor
K_ϵ = plastic strain concentration factor
m = Ramberg-Osgood exponent
R = plastic stress concentration reduction factor
R^* = reduction factor for discontinuity stress in excess of secant-yield
α = end-strain ratio
ϵ = strain
ϵ_n = discontinuity strain
ϵ_0 = reference strain
ϵ_I = secant-yield strain
ϵ_α = end-strain of Ramberg-Osgood equation
σ = stress
σ_n = discontinuity stress
σ_p = effective proportional limit stress
σ_0 = reference stress
σ_I = secant-yield stress
σ_α = end-stress of Ramberg-Osgood equation

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*Research Mechanical Engineer, Mechanics Research Laboratory. Member AIAA.

Background

SPECIFIC methods for deriving plastic concentration factors were recently described using graphic[1] and analytic[2] methods. In each case the Neuber plastic concentration factor equation[3] was used in conjunction with an analytical approximation of the stress-strain curve. In Ref. 2 results were developed using a two-piece approximation of the stress-strain curve: the Ramberg-Osgood equation[4] up to the secant-yield stress and a simple power law thereafter.[5] The rationale for the piecewise formulation was that the curve-fitting procedure for the Ramberg-Osgood equation required experimental data only up to the secant-yield stress; hence there was no assurance that the equation was suitable beyond this limit. The possibility exits, however, that a statistically satisfactory fit of experimental stress-strain data to the Ramberg-Osgood equation can be achieved for values in excess of the secant-yield stress for some materials; hence, a piecewise approximation with its added complexity may not be necessary.

It is the purpose of this Note to extend the analysis of Ref. 2, using only the Ramberg-Osgood equation, by developing relations between reference and discontinuity stresses in a form that may be useful in design, and from which nondimensional design graphs may be constructed. Included in the development is a method of specifying an end-point value for the Ramberg-Osgood equation when a satisfactory fit of experimental data can be made for stress and strain values beyond the secant-yield stress.

Ramberg-Osgood Equation

Expressed in stress-ratio form, the Ramberg-Osgood equation is:

$$(\sigma/\sigma_I) + (3/7)(\sigma/\sigma_I)^m = \epsilon/(\sigma_I/E) \tag{1}$$

Note that the quantity on the right-hand side of Eq. (1) is the ratio of a total strain value to the elastic component of the secant-yield strain. This ratio has the value 10/7 when $\sigma = \sigma_I$. When the test data indicate a fit beyond σ_I a constant α can be specified that is related to the end-point strain ϵ_α by

$$\epsilon_\alpha/(\sigma_I/E) = (10/7)\alpha \tag{2}$$

The implication of Eq. (2) is that the total strain ϵ_α for a stress σ_α is α times the total strain associated with the secant-yield stress. The end-stress value σ_α can be found when α is specified by a numerical solution of Eq. (1). This equation has the form

$$X^a + (3/7)X^b - A = 0 \tag{3}$$

where $X = \sigma_\alpha/\sigma_I$, $A = (10/7)\alpha$, $a = 1$, and $b = m$. As previously indicated[2] a solution of Eq. (3) is easily accomplished with a desk calculator or by machine computation using a simple root-finder routine.

An effective proportional limit stress is considered to be the lower limit to plastic discontinuity behavior. A nondimensional proportional limit is defined here as that value of σ/σ_l in Eq. (1) such that the term $(3/7)(\sigma/\sigma_l)^m$ is only a negligible fraction of the term σ/σ_l. Arbitrarily assuming that the ratio of the two terms is $m\times 10^{-5}$ results in an effective proportional limit value of

$$\sigma_p/\sigma_l=[(7/3)m\times 10^{-5}]^{1/m-1} \tag{4}$$

Neuber Equation

The relation between elastic and plastic concentration factors when stated in terms of stresses and strains was shown to have the form[2]

$$\sigma_0\epsilon_0=\sigma_n\epsilon_n/K_t^2 \tag{5}$$

It is assumed in a design problem that both the reference stress and the elastic concentration factor are known quantities. The initial problem in design is to relate the discontinuity stress to the reference stress.

To evaluate Eq. (5) it is necessary to construct product terms of the form $\sigma\epsilon$. By suitable algebraic manipulation of Eq. (1), it can easily be shown that

$$\sigma\epsilon=(\sigma_l^2/E)[(\sigma/\sigma_l)^2+(3/7)(\sigma/\sigma_l)^{m+1}] \tag{6}$$

Using the appropriate subscripts and substituting expressions of the form of Eq. (6) into Eq. (5) results in

$$(\sigma_n/\sigma_l)^2+(3/7)(\sigma_n/\sigma_l)^{m+1}-K_t^2[(\sigma_0/\sigma_l)^2+(3/7)(\sigma_0/\sigma_l)^{m+1}]=0 \tag{7}$$

Since (σ_0/σ_l) and K_t are assumed to be known quantities, Eq. (7) has the same form as Eq. (3) where now $X=(\sigma_n/\sigma_l)$, $A=K_t^2[(\sigma_0/\sigma_l)^2+(3/7)(\sigma_0/\sigma_l)^{m+1}]$, $a=2$, and $b=m+1$. If the value of the reference stress is at or below the effective proportional limit, Eq. (7) can be simplified to

$$(\sigma_n/\sigma_l)^2+(3/7)(\sigma_n/\sigma_l)^{m+1}-(K_t\sigma_0/\sigma_l)^2=0 \tag{8}$$

The same numerical scheme, used previously for solving Eq. (1), may also be used for solving Eqs. (7) and (8).

Stress Concentration Reduction

The reduction of the stress concentration factor from its elastic value, as the discontinuity stress increases into the plastic range, may be quantified by the use of a plastic reduction factor R defined as

$$R=K_\sigma/K_t \tag{9}$$

or in terms of stress values as

$$R=(\sigma_n/\sigma_l)/(\sigma_0/\sigma_l)K_t \tag{10}$$

If now the reduction factor is introduced into Eq. (7), the discontinuity stress can be eliminated and a relation between the reference stress and the reduction factor can be established in the form

$$(\sigma_0/\sigma_l)=[(7/3)(1-R^2)/(K_t^{m-1}R^{m+1}-1)]^{1/m-1} \tag{11}$$

For the special case where the reference stress is at or below the proportional limit, the relation becomes

$$(\sigma_0/\sigma_l)=[(7/3)(1-R^2)]^{1/m-1}/K_tR^{m+1/m-1} \tag{12}$$

As will subsequently be shown, design graphs may be constructed, using Eqs. (11) and (12), for specific values of the Ramberg-Osgood exponent.

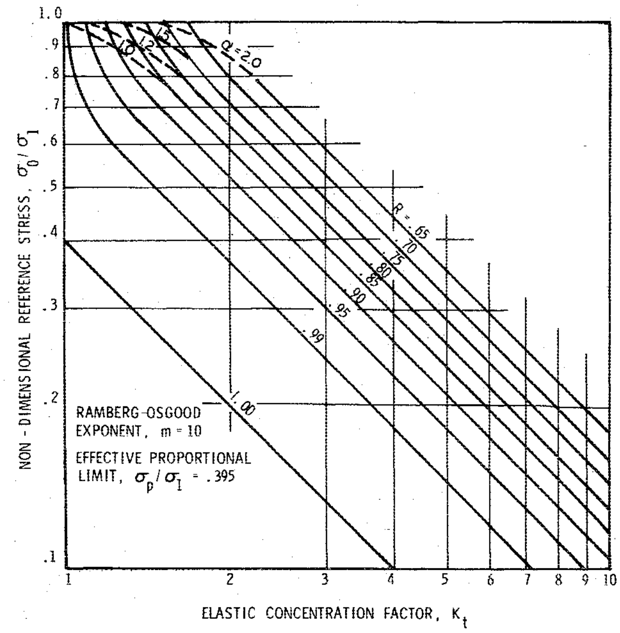

Fig. 1 Design values of the stress concentration reduction factor for a material with Ramberg-Osgood exponent $m=10$. Cutoff limits for various α values are shown as dashed curves.

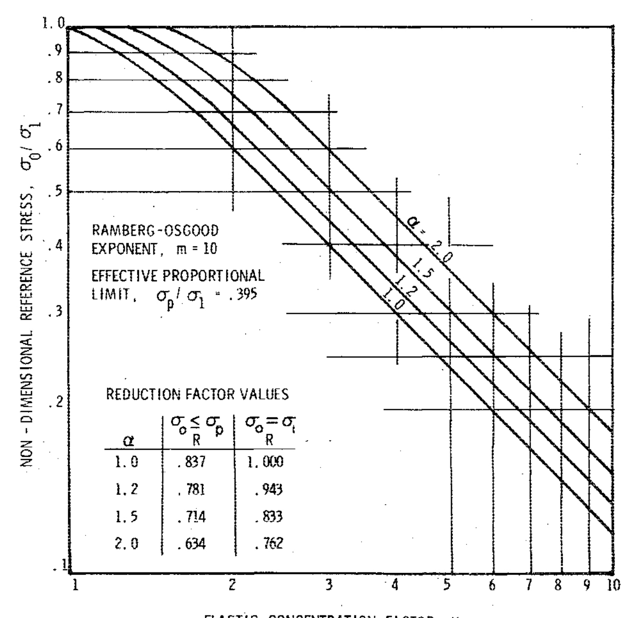

Fig. 2 Reference stress and elastic concentration factor for discontinuity stress at Ramberg-Osgood end-value. Ramberg-Osgood exponent, $m=10$.

The lower limit of the reduction factor occurs when the discontinuity stress reaches the end-point value of the Ramberg-Osgood equation. The lower-limit value is a minimum when the elastic concentration factor is sufficiently high so that the reference stress is elastic. If it is assumed that the reference stress is at the proportional limit and the discontinuity stress is at the end-value, the lower limit value of the elastic concentration factor K^* can be found from

$$K^*=[(10/7)\alpha(\sigma_\alpha/\sigma_l)]^{1/2}/[(7/3)m\times 10^{-5}]^{1/m-1} \tag{13}$$

The minimum vlaue of the reduction factor R^*, which occurs when the discontinuity stress is at the end-value and when K_t is equal to or greater than K^*, can be found by substituting Eq. (10) into Eq. (8), resulting in

$$R^* = [0.7(\sigma_\alpha/\sigma_l)/\alpha]^{1/2} \tag{14}$$

Note that the value of R^* is independent of K_t. For $\alpha = 1$, the condition when $\sigma_n = \sigma_l$, $R^* = \sqrt{0.7} = 0.8367$ for all values of the exponent m. For values of σ_n greater than σ_l, R^* is relatively insensitive to m as compared with its sensitivity to α.

Design Graphs

A sample design graph for a material whose Ramberg-Osgood exponent, $m = 10$, is shown in Fig. 1. Other similar graphs could be constructed for a range of m values. The graph allows a rapid determination of one of the three variables (σ_0/σ_l), K_t, or R, when two are specified. As an adjunct to the data in Fig. 1, the variation of σ_0/σ_l and K_t for constant values of α is shown in Fig. 2. Each curve on Fig. 2 represents the condition for a constant value of the discontinuity stress equal to the indicated Ramberg-Osgood end-value. If the reference stress is elastic, the associated reduction factor will be a constant whose value is a function of the end-strain ratio α given by Eq. (14). For reference stress in excess of the effective proportional limit, the reduction factor increases. Values of the reduction factor for all elastic reference stresses and for the reference stress at the secant-yield stress are given in Fig. 2.

A computer program has been written for calculation of R, σ_0/σ_l, σ_n/σ_l, and K_σ for a series of elastic concentration-factor values from 1.05 to 10 and for values of the Ramberg-Osgood exponent from 5 to 200. The data for Figs. 1 and 2 were obtained from the tabular output of the program.

Conclusions

1) Design relations have been developed for plastic discontinuity stresses using Neuber's equation and the Ramberg-Osgood analytic approximation of stress-strain properties.

2) The limiting value of the discontinuity stress in the developed relations is the Ramberg-Osgood end-value. The latter is the maximum value of stress for which the Ramberg-Osgood equation provides a satisfactory fit of experimental stress-strain data. The end-point of the equation has been identified using a non-dimensional strain parameter. The end-stress is obtained by computation.

3) The relative reduction of the stress concentration factor associated with plastic discontinuity stresses is independent of the elastic concentration factor. The reduction is mainly dependent on the maximum discontinuity stress and the Ramberg-Osgood exponent value.

4) Limiting values of the stress-concentration reduction factor occur when the discontinuity stress is at the Ramberg-Osgood end-value and the reference stress is elastic: a) If the end-value is the secant-yield stress, the plastic concentration factor is 83.7% of the elastic concentration factor for all values of the Ramberg-Osgood exponent and for all elastic values of the reference stress; or b) If the end value is greater than the secant-yield stress, the reduction value is a function of both the exponent and the end-stress. However, for given values of the two parameters, the limiting value of the reduction factor is constant for all elastic values of the reference stress.

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Some Singular Aspects of Three-Dimensional Transonic Flow

Wilson C. Chin*
Boeing Commercial Airplane Company, Seattle, Wash.

Introduction

THIS Note discusses certain singular aspects in the steady formulation for three-dimensional transonic flows. The method of inner and outer expansions is used to show how two separate limits can be distinguished for the inner crossflow problem, the first leading to Laplace's equation and the second leading to a mixed-type equation. The particular equation used in any problem is fixed by the relative value of an aspect ratio A to a measure ϵ of the flow nonlinearity. The required matching process then determines the role of the outer small disturbance equation (SDE) in calculating near-field surface pressures.

Analysis

The classical SDE for steady three-dimensional transonic flow is useful in describing the near-isentropic flow about only certain types of thin wings. The most severe constraint appears to be that for near two dimensionality, that is, $A\tau^{1/3} \gg 1$, where τ is the thickness ratio. For slender, highly swept wings with $A\tau^{1/3} \ll 1$, the SDE does not seem to produce very encouraging results. For example, the three-shock pattern observed experimentally on swept supercritical wings (consisting of a conical forward shock, a rear shock, and a tip shock) cannot be predicted. This situation has forced a number of authors to reassess the usefulness of the SDE and a number of heuristic corrections have been proposed,[1,2] these models typically adding terms that contain various quadratic and cubic nonlinearities, spanwise terms, and so on. The most critical step implicit in these schemes assumes that the new equation describes *both* "inner" near-field and "outer" far-field flows. This is certainly not clear a priori since the problem must be studied using the complete equation. The failure of classical theory for certain configurations suggests that A must be included in the transonic limiting process. In the classical derivation[3] all space coordinates are normalized by the same reference length; only after the SDE is obtained are the well-known similarity rules involving A obtained. Thus, the approach taken here follows the spirit in which constant density slender body and planar wing theories are derived from the three-dimensional Laplace equation. The full potential equation is considered with A included at the outset, and different near-field equations are derived corresponding to different limiting processes.

The exact dimensional equation for the disturbance potential $\varphi(x,y,z)$ contains quadratic and cubic nonlinearities that may not be negligible. Let M_∞ denote the subsonic freestream Mach number, U_∞ the freestream speed, γ the ratio of specific heats; the streamwise, spanwise, and normal coordinates here being x,y, and z. Barred nondimensional

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*Specialist Engineer, Aerodynamics Research Group. Member AIAA.